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Imitation, Network Size, and Efficiency*

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Abstract

A number of theoretical results have provided sufficient conditions for the selection of payoff-efficient equilibria in games played on networks when agents imitate successful neighbors and make occasional mistakes (stochastic stability). However, those results only guarantee full convergence in the long-run, which might be too restrictive in reality. Here we employ a more gradual approach relying on agent-based simulations avoiding the double-limit underlying these analytical results. We focus on the circular-city model, for which a sufficient condition on the population size relative to the neighborhood size was identified in Alós-Ferrer and Weidenholzer (2006). Using more than 100,000 agent-based simulations, we find that selection of the efficient equilibrium prevails also for a large set of parameters violating the previously identified condition. Interestingly, the extent to which efficiency obtains decreases gradually as one moves away from the boundary of this condition.

JEL Classification: C63 · C72 · C73

Keywords: agent-based models; pareto efficiency; risk dominance; imitation; networks; stochastic stability

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1 Introduction

One of the milestones of the literature on learning in games is the result that risk-dominant equilibria are selected in the long run (even in the presence of alternative, Pareto-efficient ones) when agents follow simple behavioral rules and make mistakes (Kandori et al., 1993). This result, however, crucially depends on the assumption that every player interacts with every other player in the population (global interactions). Starting with the seminal paper of Ellison (1993) on best-reply behavioral rules and the circular-city model, the importance of the interaction structure (often referred to as network) has received considerable attention. Ellison (2000) and Morris (2000) examined equilibrium selection in networks when agents follow myopic best reply, and found results in line with the selection of risk-dominant equilibria or suitable generalizations, although other work (Alós-Ferrer and Weidenholzer, 2007) has shown that this basic message cannot be generalized beyond the case of 2×2 games.

Since risk-dominant equilibria are not necessarily payoff-dominant, those results show that best-reply rules, which focus only on individual maximization, can generally result in inefficient outcomes. In contrast, a number of results have shown that, if agents follow imitation rules, which are comparatively simpler but use payoff information from other agents, payoff-dominant equilibria play a more important role. Robson and Vega-Redondo (1996) showed that, if agents imitate best observed payoffs but interact with only one randomly sampled other agent per period (instead of focusing on average payoffs across a large number of interactions as in Kandori et al., 1993), payoff-dominant equilibria are selected. Even though this is a global-interactions model, the result already highlights the importance of the reach of interactions for equilibrium selection. Alós-Ferrer and Weidenholzer (2008) showed that, if agents have access to information slightly beyond their interaction neighborhoods, payoff-dominant equilibria are selected in arbitrary networks as long as the network size is large enough relative to the (smallest) size of neighborhoods, a result which survives independently of the size of the game. A similar result was obtained for minimum-effort games in Alós-Ferrer and Weidenholzer (2014).[1]

In this work, we concentrate on imitation rules in the absence of informational spillovers, a case which, surprisingly, is still far from settled. Alós-Ferrer and Weidenholzer (2006) studied agents using an imitate-the-best rule playing a 2×2 coordination game under local information and asked whether risk-dominant or efficient conventions would be selected. The answer depends on the interaction radius of the agents, i.e. on "how local" interactions are. Specifically, that work considers a circular city and shows that the risk-dominant outcome is the unique long-run equilibrium if agents only interact

[1] Payoff-dominant equilibria can also be selected in some models of interaction through best-reply rules, but, intuitively speaking, only if an additional dimension of behavior allows to avoid undesirable interactions. For instance, Goyal and Vega-Redondo (2005) studied a model of endogenous network formation where players can build and severe costly links, and showed that coordination on the payoff-dominant one occurs if links are costly enough.

with their direct neighbors. However, if their interaction radius is larger, the efficient equilibrium is uniquely selected if the population is relatively large compared to the interaction radius (a condition conceptually aligned with the results in Alós-Ferrer and Weidenholzer, 2008, 2014).[2]

The circular city remains a theoretically important benchmark, where the interaction radius neatly captures the idea that interactions can be neither purely local nor completely global. In this setting, the result of Alós-Ferrer and Weidenholzer (2006) remains the theoretical state of the art, but it has two limitations that we address here. First, it identifies a sufficient condition, and hence the question of whether payoff-efficient or risk-dominant outcomes are selected when that condition is violated remains open. Second, the selection result, as standard in the stochastic stability literature, only addresses convergence in the long-run, which entails a double limit with vanishing mistakes as time goes to infinity. As a consequence, the identified conditions might be unrealistically restrictive, and the result refers only to full convergence. Here we adopt a gradual approach based on extensive agent-based simulations, as in, e.g., Alós-Ferrer and Buckenmaier (2017) and Khan et al. (2016). Our computational results show that the extent to which convergence obtains exhibits no sharp discontinuity at the boundary describing the previously-identified, sufficient condition, but instead changes continuously. The intuition for this result is simple. As the relative network size decreases, crossing the theoretical boundary from above in the parameter space, transitions toward the payoff-efficient equilibrium become increasingly difficult, whereas transitions toward the risk-dominant equilibrium become easier. Despite this continuity, we find that the process spends relatively more time at the payoff-efficient equilibrium. In this sense, our results show that "selection" of the efficient equilibrium obtains for a larger set of parameters than that covered by the theoretical result.

The joint parameter space describing games, networks, and behavioral noise is staggeringly large. A full-factorial approach is not feasible in practice at this point. Hence, we conduct two types of simulations. In Study 1, we run 60,000 simulations with a *random sampling approach*, where the parameters of each simulation are randomly drawn from the full space with some suitable distributional conditions. In Study 2, and as a robustness check, we run a factorial design for a reduced number of values (covering 36 games), for a total of 44,100 additional simulations. The results of both studies are in agreement.

Section 2 below describes the setting. Section 3 describes the main set of simulations (Study 1) and the results of the analysis. Section 4 presents additional robustness analyses, including the results of Study 2. Section 5 concludes.

[2]The model of Alós-Ferrer and Weidenholzer (2006) was revisited by Chen et al. (2012) for the case where risk-dominant equilibria are *also* payoff-dominant. The authors find that risk-dominant but also other equilibria can be selected when players interact with their immediate neighbors only.

2 The model

The notation follows Alós-Ferrer and Weidenholzer (2006, 2008). Consider the classical problem of selection between a risk-dominant (R) and a Pareto-efficient (P) action in a 2×2 coordination game with payoff function π given by

	P	R
P	$1,1$	$0,\alpha$
R	$\alpha,0$	β,β

where $0 < \alpha < 1$ and $\beta < 1$ so that both (P,P) and (R,R) are strict Nash equilibria. Further, assume that $\alpha + \beta > 1$, that is, (R,R) is *risk dominant*.

There is a population of N players arranged on a circle so that the immediate neighbors of a player i are $i-1$ and $i+1$ (modulo N). Players play the game against their $2k$ closest neighbors with $k \in \{1,\ldots,\lfloor \frac{N}{2} \rfloor\}$. The neighborhood of player i is $K(i) = \{i-k,\ldots,i-1,i+1,\ldots,i+k\}$ for $k \neq \lfloor \frac{N}{2} \rfloor$ and $K(i) = N \setminus \{i\}$ for $k = \lfloor \frac{N}{2} \rfloor$. The latter corresponds to global interactions. Given a strategy profile $\omega = (s_i)_{i \in N}$ the payoff of player i is $\Pi_i(\omega) = \frac{1}{|K(i)|} \sum_{j \in K(i)} \pi(s_i, s_j)$. Players observe the actions and payoffs received by themselves and their neighbors. They follow an imitate-the-best rule with player i choosing a strategy in the set

$$B_i(\omega) = \{s_i' \mid s_i' = s_j \text{ with } j \in K(i) \cup \{i\},\ \Pi_j(\omega) \geq \Pi_l(\omega)\ \forall l \in K(i) \cup \{i\}\}.$$

This defines a dynamic adjustment process described by a Markov process in discrete time. In addition, players might make mistakes with a given probability $\varepsilon > 0$. The stochastically stable states are the long-run equilibria of the process as noise vanishes ($\varepsilon \longrightarrow 0$).

Alós-Ferrer and Weidenholzer (2006, Theorem 1) show that for $k = 1$ the risk-dominant outcome is the unique long-run equilibrium. In contrast, their main result (Theorem 2) states that for $k > 1$ the efficient equilibrium is uniquely selected if the population is relatively large compared to the size of neighborhoods, provided neighborhoods are not too small.

Theorem 2 (Alós-Ferrer and Weidenholzer, 2006)**.** *In the $2k$-neighbor model coordination on the efficient strategy is the unique long-run equilibrium under the imitate-the-best rule whenever*

$$N > 2k(2k+1) \tag{1}$$

provided $k \geq \frac{1}{2-2\alpha}$ (if $\alpha > \beta$) or $k \geq \frac{1}{2-(\alpha+\beta)}$ (if $\alpha \leq \beta$).

In this work we rely on extensive simulations to study the tightness of the "large population" condition (1) in a setting with small but non-vanishing noise. Our analysis will also show the extent to which convergence obtains as one approaches the boundary of this condition.

Table 1: Simulation parameters, Study 1.

Type of Simulation	Parameter	Values
All	N	$N \sim U(\{4, \ldots, 100\})$
	k	$k \sim \min\{\gamma(2, N/16), \lfloor \frac{N}{2} \rfloor\}$
	ε	$\varepsilon \sim U([0.01, 0.1])$
$\alpha > \beta$	α	$\alpha \sim U((\frac{1}{2}, \frac{3}{4}])$
	β	$\beta \sim U((\frac{1}{2}, \alpha))$
$\alpha \leq \beta$	β	$\beta \sim U((\frac{1}{2}, \frac{3}{4}])$
	α	$\alpha \sim U([\frac{1}{2}, \beta])$

Notes: $U(S)$ denotes the uniform distribution over $S \subseteq \mathbb{R}$. $\gamma(a, b)$ denotes the gamma distribution with shape parameter a and rate parameter b. 60,000 simulations were conducted, with randomly-sampled parameter values.

3 Study 1

The relevant space that we intend to explore is five-dimensional. Networks are characterized by the network size N and the neighborhood size k. Underlying games are characterized by the payoff parameters α, β. Further, avoiding the double-limit approach adds the mutation probability ε as a further parameter. Simple computations show that a truly exhaustive computational analysis of the state space remains unpractical. Hence, in a first study, we adopt a *random sampling approach.* Each simulation used a set of randomly-drawn parameters according to Table 1. We conducted $60,000$ simulations, half of them with $\alpha > \beta$ and half with $\alpha \leq \beta$. In both cases the range of α and β was chosen such that for $k > 1$ the second condition of Theorem 2 is satisfied for all feasible combination of α and β. Further, note that for the chosen range any state with a cluster of at least $2k$ adjacent players choosing the risk-dominant action lies in the basin of attraction of R (see Alós-Ferrer and Weidenholzer, 2006, Lemma 2). That is, $2k$ simultaneous mutations are sufficient to leave the state were all players choose P.

3.1 Description of the Simulations

The dynamics described above yields an agent-based simulation protocol as follows. At time $t = 0$, we start with a random fraction and distribution of P-players. The initial profile $\omega_0 = (s_i)_{i \in N}$ is obtained assigning $s_i = P$ or R with probability 0.5 each, independently for each i. The payoff of player i in period t is $\Pi_i(\omega_t)$, which determines the set of "best-performing strategies" $B_i(\omega_t)$ observed by i. Each player adopts a strategy $s_i \in B_i(\omega_t)$, breaking ties randomly if needed. Further, with a fixed probability ε, player i makes a mistake and chooses $s_i' \neq s_i$ instead of s_i (independently drawn for each player). This process is then repeated for $T = 200,000$ periods.

3.2 Results

The objective of the simulations was to obtain estimates of the (limit) invariant distribution μ^*, which determines the long-run equilibria of the dynamics (see, e.g., Ellison, 2000). We rely on the Ergodic Theorem (Karlin and Taylor, 1975) to obtain an approximation of μ^* through the average time spent by the system in a given state for (long enough) simulations (as in Alós-Ferrer and Buckenmaier, 2017). Formally, μ^* is a distribution over the set of absorbing states $\Omega \subseteq \{P, R\}^N$. The monomorphic states $\bar{P}$ and $\bar{R}$ in which all players choose P or R, respectively, are always absorbing for imitation-based dynamics. However, there might be other, non-monomorphic, absorbing states. Since we are interested in whether coordination on P or R occurs, we focus on the two monomorphic states and aggregate all non-monomorphic states (absorbing or not) in a residual, denoted *res*. Specifically, for each simulation we record the fraction of time spent in each monomorphic state. Our main output is then the relative probability distribution $f : \{\bar{P}, \bar{R}\} \cup \{res\} \longrightarrow [0, 1]$. To control for mechanical differences in the amount of time spent in non-monomorphic states (e.g. due to different values of ϵ relative to N), we focus on the restriction of the relative frequency distribution to $\{\bar{P}, \bar{R}\}$, which we denote by $F : \{\bar{P}, \bar{R}\} \longrightarrow [0, 1]$. That is, we consider the relative frequency of monomorphic states normalized by the joint frequency of all monomorphic states.

Alós-Ferrer and Weidenholzer (2006) study selection by analytically identifying the support of the limit invariant distribution. In contrast, we consider selection in a numerical sense. Specifically, we say that P is (numerically) *selected* if (empirically) the process spends relatively more time at $\bar{P}$ than at $\bar{R}$, that is, if $F(\bar{P}) \geq F(\bar{R})$. Selection of R is defined analogously.

Figure 1 shows a heatmap of the data set in the (k, N)-parameter space. The average value of $F(\bar{P})$ for each (k, N) pair is color-coded. The dashed line depicts the boundary given by condition (1). Blank cells are those where there is no data (which is possible due to random sampling). The figure illustrates that for $k = 1$ full coordination on R obtains independently of network size (Alós-Ferrer and Weidenholzer, 2006, Theorem 1). Thus, we empirically confirm the theoretical prediction that for $k = 1$ the unique long-run equilibrium is given by R. Since this result is independent of the network size N, it serves as an important first validation of our approach. For $k > 2$, in the area where (1) holds (left of the dashed line), we observe full coordination on P as predicted by Theorem 2. Again, our results numerically confirm the analytical selection of P in that area, which provides a second validation that our concept of (numerical) selection in terms of the relative frequency mirrors the known theoretical results. Interestingly, the average value of $F(\bar{P})$ is close to 1 for a relatively large area where (1) is violated (right of the dashed line). Thus, the simulation results indicate that the efficient equilibrium is selected for a large range of parameters where Theorem 2 remains silent. Generally, there is no sharp boundary separating high and low average values of $F(\bar{P})$, but instead we observe a rather continuous (but somewhat noisy) transition from full coordination

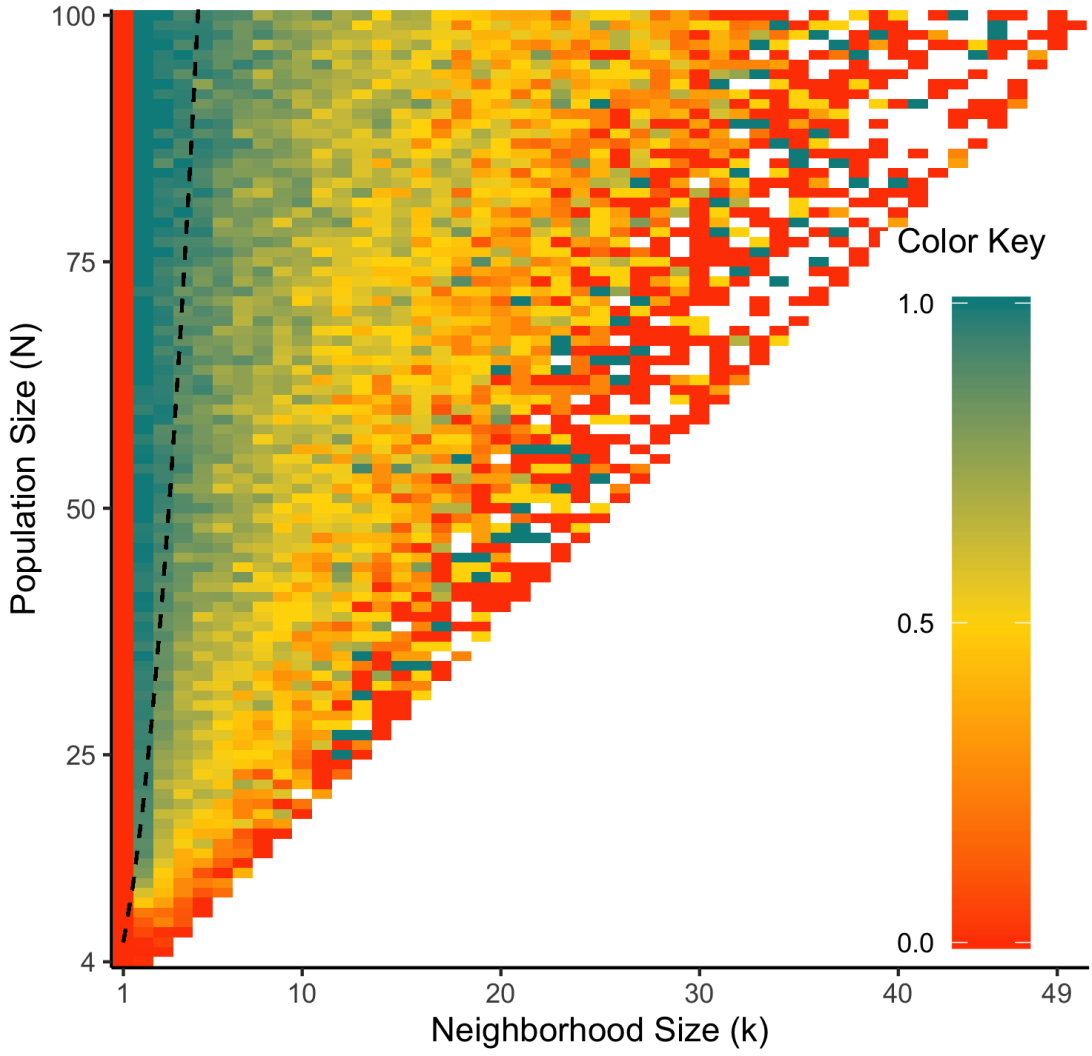

Figure 1: Heatmap of average value of $F(\bar{P})$, Study 1. Color code indicates value of $F(\bar{P})$ for each (k, N) pair. Dashed line corresponds to condition (1).

on the efficient equilibrium to full coordination on the risk-dominant one being more frequent. This insight cannot be obtained from a standard stochastic-stability analysis, which only yields binary predictions (whether a state is a long-run equilibrium or not).

Intuitively, this gradual relation should not come as a surprise. The Radius-Coradius result underlying Theorem 2 implies that leaving the basin of attraction of $\bar{P}$ requires at least $\frac{N}{2k+1}$ mutations, whereas the basin of attraction of $\bar{R}$ can be left with no more than $2k$ mutations. Hence, it is likely that even below but very close to the boundary given by $N = 2k(2k+1)$, most transitions from $\bar{P}$ to $\bar{R}$ are still more difficult than most transitions in the opposite direction. However, the further we move away from this boundary the more the balance shifts, with $\bar{R}$ eventually being selected more frequently than $\bar{P}$ as N becomes smaller relative to $2k(2k+1)$. Following this logic, intuitively one would expect $F(\bar{P})$ to be increasing in N and decreasing in k.

Table 2: Fractional logit regressions for $k > 1$, Study 1.

$F(\bar{P})$	1	2	3	4
LP	2.6196***	1.7737***	3.9950***	1.5748***
	(0.0542)	(0.0561)	(0.1959)	(0.0796)
k		-0.1430***	-0.1438***	
		(0.0026)	(0.0026)	
N		0.0252***	0.0256***	
		(0.0005)	(0.0005)	
$LP \times k$			-1.1856***	
			(0.1195)	
$LP \times N$			0.0169**	
			(0.0045)	
$\lvert N - 2k(2k+1)\rvert$				-0.0007***
				(0.0000)
$LP \times \lvert N - 2k(2k+1)\rvert$				0.0406***
				(0.0042)
ε	1.7659***	2.0358***	2.0233***	1.8994***
	(0.3728)	(0.3919)	(0.3925)	(0.3815)
Constant	0.4981***	0.3739***	0.3610***	0.7817***
	(0.0227)	(0.0316)	(0.0316)	(0.0243)
Observations	52909	52909	52909	52909

Notes: Standard errors in parentheses. * $p < 0.01$, ** $p < 0.001$, *** $p < 0.0001$.

To confirm these observations and to further study the role of N and k, we turn to a regression analysis. Since $F(\bar{P})$ is a frequency, we use fractional logit regressions (Papke and Wooldridge, 2008). Table 2 shows the results of these regressions. For $k = 1$ full coordination on R obtains and no gradual effects are expected, hence those observations are excluded from the regression. As independent variables we include k, N, ε, and a dummy LP indicating that (1) is fulfilled, that is, $LP = 1$ if $N > 2k(2k + 1)$, and zero otherwise.

Model 1 shows that violating (1) significantly decreases the weight on the efficient equilibrium. In model 2, we include the variables k and N to study how the weight on $\bar{P}$ changes as one moves horizontally or vertically, respectively. In line with the general intuition that increasing k relative to N makes the efficient equilibrium more difficult to establish, we find a negative effect of k on $F(\bar{P})$ and a positive effect of population size. That is, coordination on P becomes gradually more difficult as population size decreases and neighborhood size increases. Model 3 shows that this result is robust when we include the interactions of k and N with the dummy LP. The direction of the effect of an increase in N or k within the area covered by Theorem 2 ($LP = 1$), i.e. as one moves toward the boundary of (1), is the same. In view of Theorem 2, the interpretation is that long-run convergence takes longer to obtain as population size decreases and neighborhood size increases. Finally, we want to link the observed gradual effects of N and k directly to the tightness of condition (1). To that end, we consider the variable $|N - 2k(2k + 1)|$,

which continuously captures the extent to which this condition is satisfied or violated. Model 4 includes this continuous variable as well as its interaction with *LP*. When (1) is violated, moving away from the boundary has a negative effect, whereas the effect is positive when (1) is satisfied. That is, in line with the above intuition, we find that the relative weight of the payoff-efficient outcome decreases the smaller N becomes relative to $2k(2k+1)$. Finally, we observe a somewhat surprising positive effect of the mutation probability ε in all specifications. We will return to this point in Section 4.1 below.

4 Robustness Analysis

4.1 Residual and non-monomorphic absorbing states

In the previous section, our analysis relied on the restriction of the relative frequency distribution to the set of monomorphic states $\{\bar{P}, \bar{R}\}$. However, in a setting with non-vanishing mutation probability as the one we consider, it is to be expected that the process also spends a certain amount of time in other absorbing or even non-absorbing states. To address this issue, we now report several robustness checks using the non-restricted relative frequency distribution $f : \{\bar{P}, \bar{R}\} \cup \{res\} \longrightarrow [0, 1]$.

We first compare the time spent in monomorphic states relative to the time spent in other non-monomorphic states. The latter is captured by the weight on the residual $f(res)$ and contains both absorbing non-monomorphic states and non-absorbing states. Across all simulations the dynamic spends 36.2% of the time in one of the two monomorphic states $\bar{P}$ or $\bar{R}$. Naturally, the process should spend less time overall in non-absorbing states the smaller ϵ is. To test for such a relation, we ran an additional fractional logit regression with dependent variable $f(res)$ that controls for the mutation probability ε. We find a significant positive effect of ε on the relative weight of the residual ($\beta = 28.8343$, SE: 0.1813, $p < 0.0001$). Indeed, for the bottom 10% of values of ε the weight on the residual is on average only 29.9%, whereas it reaches 82.4% on average for the top 10% of mutation probabilities.

As a further robustness check, we ran analogous regressions to the ones reported in Table 2 but using the non-restricted relative frequency distribution f instead of F. Note that for the former distribution the relative weight of $\bar{R}$ is not simply given by $1 - f(\bar{P})$. Hence, we report two sets of regressions, one with $f(\bar{P})$ and one with $f(\bar{R})$ as dependent variable. Table 3 shows the results of these regressions. Models 1–3 show that the effects of the dummy *LP* as well as the effect of the neighborhood size k are qualitatively the same as the ones reported in Table 2. However, the effect of the network size N in models 2 and 3 is negative, whereas we observed a positive effect of N previously. Note that since the probability of a mutation is independent across agents, the overall likelihood of a *single* mutation is not only increasing in ε but also in N. That is, for larger network size more weight is shifted to the residual. Hence, in model 4 we additionally control for the overall frequency of mutations (number of actual mutations in the simulation divided

Table 3: Fractional logit regressions for $k > 1$, Study 1.

	$f(\bar{P})$ (models 1–4)				$f(\bar{R})$ (models 5–8)			
	1	2	3	4	5	6	7	8
LP	0.4780***	0.1868***	0.8149***	0.8365***	-2.1645***	-1.6748***	-3.4935***	-3.3220***
	(0.0114)	(0.0120)	(0.0328)	(0.0279)	(0.0626)	(0.0631)	(0.2562)	(0.2559)
k		-0.0706***	-0.0795***	-0.0813***		0.0908***	0.0915***	0.0959***
		(0.0017)	(0.0018)	(0.0018)		(0.0015)	(0.0015)	(0.0016)
N		-0.0167***	-0.0139***	0.0102***		-0.0478***	-0.0481***	-0.0231***
		(0.0003)	(0.0003)	(0.0006)		(0.0006)	(0.0006)	(0.0009)
$LP \times k$			0.0313	0.0280*			1.0103***	1.0016***
			(0.0126)	(0.0098)			(0.1635)	(0.1638)
$LP \times N$			-0.0144***	-0.0144***			-0.0135	-0.0157*
			(0.0005)	(0.0004)			(0.0058)	(0.0057)
ε	-27.4712***	-29.4569***	-29.5362***	-4.9239***	-22.9124***	-25.1552***	-25.1760***	-1.9981*
	(0.2464)	(0.2439)	(0.2434)	(0.5470)	(0.4096)	(0.3983)	(0.3985)	(0.7638)
FreqError				-0.5522***				-0.6244***
				(0.0000)				(0.0000)
Constant	-0.0048	1.4896***	1.4176***	0.2865***	-0.8098***	0.8135***	0.8222***	-0.2075***
	(0.0141)	(0.0230)	(0.0236)	(0.0359)	(0.0221)	(0.0295)	(0.0296)	(0.0468)
Observations	52909	52909	52909	52909	52909	52909	52909	52909

Notes: Standard errors in parentheses. * $p < 0.01$, ** $p < 0.001$, *** $p < 0.0001$.

by the number of periods), which restores a positive effect of N on $f(\bar{P})$ in line with our previous result. Models 5–8 show the results of the analogous regressions that use $f(\bar{R})$ as a dependent variable. Essentially all results mirror the ones obtained for $f(\bar{P})$ but with the opposite sign. Summarizing, the robustness check confirms all our previous results, which we interpret as confirmatory evidence that the restricted relative frequency distribution F provides a meaningful summary of the long-run invariant distribution.

Finally, we would like to comment on the role of the mutation probability ε. Recall that in Table 2 we observed a somewhat surprising, positive effect of ε. However, in Table 3 we see a clear negative effect of ε on both $f(\bar{P})$ and $f(\bar{R})$, which is in line with the intuition that more noise should shift weight from absorbing states to non-absorbing states in the residual. In light of this result, we view the positive effect of ε on $F(\bar{P})$ as an indicator that an increase in the mutation probability destabilizes the risk-dominant equilibrium more that it destabilizes the payoff-efficient equilibrium.

4.2 Non-random sampling: Study 2

Our main goal was to explore the parameter range beyond the set covered by previous results. To that end, Study 1 relied on simulations with randomly-drawn parameters for each run. As a consequence, each cell in Figure 1, corresponding to a particular (N, k) combination, contains a variable number of observations, with different values of the remaining parameters α, β, and ε. In particular, cells closer to the 45-degree line contain less observations than those closer to the boundary condition (1), and some cells contained no data. To address this problem, we ran a second set of simulations following a full-factorial design for a reduced number of parameter values. Specifically, we fixed six values for each of the payoff parameters α and β, for a total of 36 games. For each (α, β)

Table 4: Simulation parameters, Study 2.

Parameter	Values
N (even)	$\{4, 6, 8, \ldots, 98, 100\})$
k	$\{1, 2, \ldots, (N/2) - 1\}$
ε	0.1
α	$\{0.5, 0.55, 0.6, 0.65, 0.7, 0.75\}$
β	$\{0.5, 0.55, 0.6, 0.65, 0.7, 0.75\}$

Notes: We use 36 combinations of α and β. For each (α, β) combination there are a total of 1225 combinations of N and k. This yields a total of $36 \cdot 1225 = 44100$ simulations.

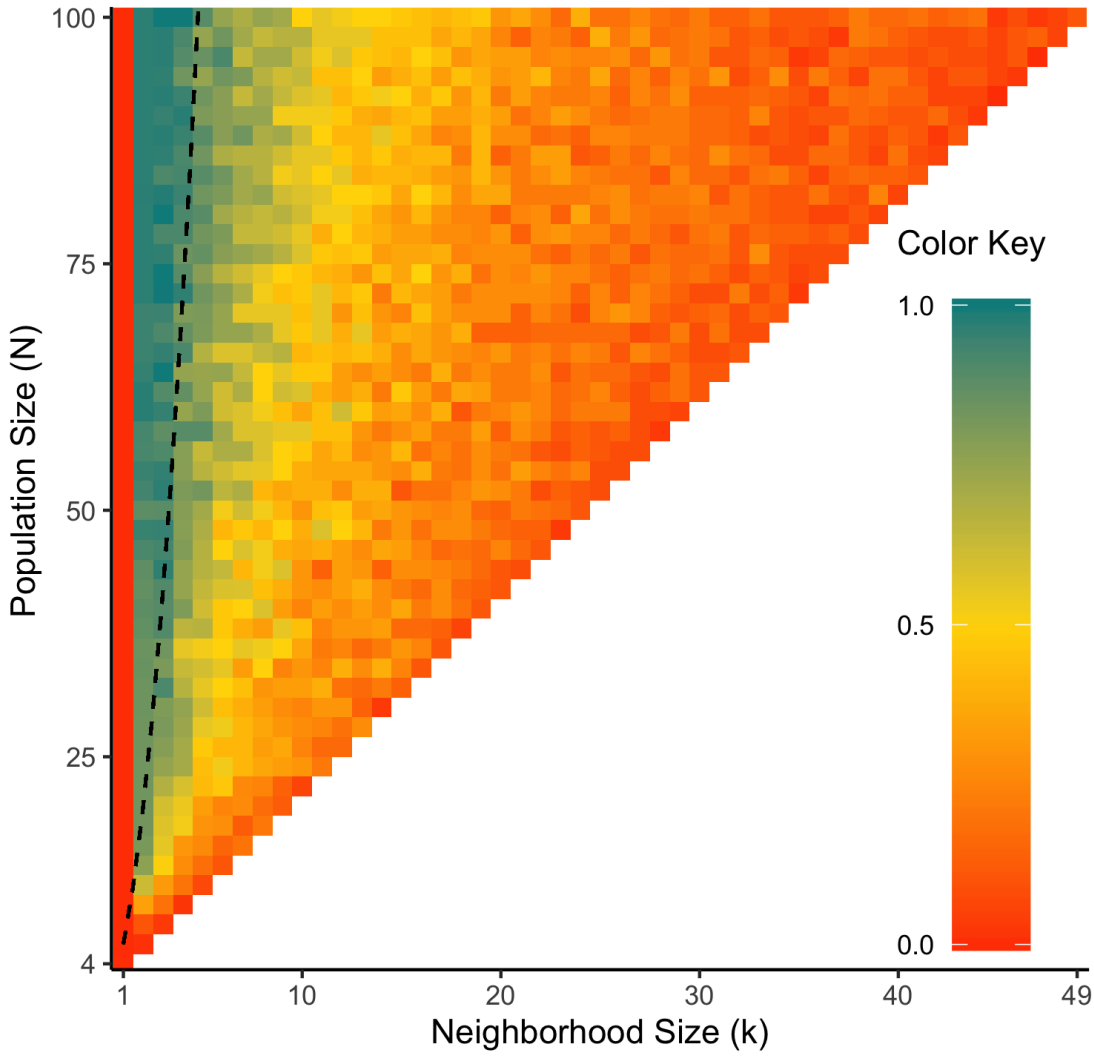

Figure 2: Heatmap, average value of $F(\bar{P})$, Study 2. Color code indicates value of $F(\bar{P})$ for each (k, N) pair. Dashed line corresponds to condition (1).

combination we then ran 1225 combinations of N and k, covering the same range of N and k as before but restricting N to even values. We fixed $\varepsilon = 0.1$ for all simulations.

Table 5: Fractional logit regression for $k > 1$, Study 2.

$F(P)$	1	2	3
LP	3.3003***	1.8134***	1.9646***
	(0.0703)	(0.0729)	(0.2998)
k		-0.1060***	-0.1059***
		(0.0016)	(0.0016)
N		0.0193***	0.0192***
		(0.0005)	(0.0006)
$LP \times k$			-0.2522
			(0.1295)
$LP \times N$			0.0085
			(0.0042)
Constant	-0.6813***	-0.1715***	-0.1669***
	(0.0106)	(0.0328)	(0.0331)
Observations	42263	42263	42263

Notes: Standard errors in parentheses. * $p < 0.01$, ** $p < 0.001$, *** $p < 0.0001$.

This resulted in 44100 new simulations. Table 4 shows the complete set of parameters used for Study 2.

Figure 2 shows the heatmap of Study 2's simulations. Each of the 1125 (N, k) cells now contains exactly 36 simulations, and all cells share the same 36 $(\alpha, \beta, \varepsilon)$ combinations. In contrast to Figure 1, by construction, there are no empty cells. The number of cells in the heatmap, however, is reduced by half since N was limited to even numbers. Again, we observe a smooth gradual pattern, which is less noisy compared to Study 1, since in Study 2 all cells contain a reasonable number of observations. Overall, the additional simulations confirm all previous results. Further, they serve as a robustness check confirming that the observed pattern is not driven by differences in the constellations of parameters other than N and k between cells. Table 5 displays the analogous regressions to those in Table 2, confirming our previous conclusions for the new set of simulations. The only difference is that the interaction terms are not significant.

5 Conclusion

We use extensive agent-based simulations (more than 100,000 simulations in two studies) to analyze the selection between risk-dominant and Pareto-efficient equilibria in coordination games played on circular-city networks. Our simulations rely on long but finite (and bounded) time horizons, and on small but positive error rates (as opposed to the double-limit approach entailed in any stochastic stability analysis). The results show that selection of the efficient equilibrium obtains for a larger range of interaction structures and network sizes than covered by previous results. Further, the degree of convergence to the efficient equilibrium changes continuously with population and neighborhood size as one moves beyond the sufficient condition previously identified.

More specifically, previous theoretical results only established sufficient conditions. If interactions are not exclusively local, payoff-dominant equilibria are selected provided a specific condition is satisfied, which ensures that the population size is large enough with respect to neighborhood size. This condition, however, covers only a small part of the parameter space. Our results show that the result extends to the entire space in a gradual way. In other words, if the interactions are not exclusively local, what determines whether imitation dynamics select predominantly payoff-dominant or risk-dominant equilibria is the population size relative to neighborhood size. As population size decreases or neighborhood size increases, the weight of risk-dominant equilibria increases gradually, compared to payoff-dominant ones. Hence, previous results were not of an "on/off" nature, but rather an indication of a continuous relationship between network size and the selection of efficient equilibria.

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